

Primitive Cohomology of Real Degree Two on Compact Symplectic Manifolds *

Qiang Tan, Hongyu Wang[†], Jiuru Zhou

Abstract: In this paper, we define the generalized Lejmi's P_J operator on a compact almost Kähler $2n$ -manifold. We get that J is C^∞ -pure and full if $\dim \ker P_J = b^2 - 1$. Additionally, we investigate the relationship between J -anti-invariant cohomology introduced by T.-J. Li and W. Zhang and new symplectic cohomologies introduced by L.-S. Tseng and S.-T. Yau on a closed symplectic 4-manifold.

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1 Introduction

For an almost complex manifold (M, J) , T.-J. Li and W. Zhang [18] introduced subgroups, H_J^+ and H_J^- , of the real degree 2 de Rham cohomology group $H_{dR}^2(M, \mathbb{R})$, as the sets of cohomology classes which can be represented by J -invariant and J -anti-invariant real 2-forms, respectively. Let us denote by h_J^+ and h_J^- the dimensions of H_J^+ and H_J^- , respectively.

It is interesting to consider whether or not the subgroups H_J^+ and H_J^- induce a direct sum decomposition of $H_{dR}^2(M, \mathbb{R})$. In the case of direct sum decomposition, J is said to be C^∞ pure and full. This is known to be true for integrable almost complex structures J which admit compatible Kähler metrics on compact manifolds of any dimension. In this case, the induced decomposition is nothing but the classical real Hodge-Dolbeault decomposition of $H_{dR}^2(M, \mathbb{R})$ (see [5]).

In dimension 4, T. Draghici, T.-J. Li and W. Zhang [9, Theorem 2.3] proved that on any closed almost complex 4-manifold (M, J) , J is C^∞ pure and full. Further in [10], they computed the subgroups H_J^+ and H_J^- and their dimensions h_J^+ and h_J^- for almost complex structures metric related to an integrable one.

In the fifth section of [16], Lejmi introduced the differential operator P_J on a compact almost Kähler 4-manifold (M, g, J, ω) ,

$$\begin{aligned} P_J : \Omega_0^2 &\rightarrow \Omega_0^2 \\ \psi &\mapsto \frac{1}{2}\Delta_g \psi - \frac{1}{4}g(\Delta_g \psi, \omega)\omega. \end{aligned}$$

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[†]E-mail: hywang@yzu.edu.cn

He proved that P_J is a self-adjoint strongly elliptic linear operator of order 2. In this paper, we define the generalized operator P_J on a compact almost Kähler $2n$ -manifold. We prove that P_J is also a self-adjoint strongly elliptic linear operator on a compact almost Kähler manifold (M, g, J, ω) of dimension $2n$.

Proposition 2.3. *Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold, then $\ker P_J = \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+$ and the harmonic representatives of H_J^- and $H_{J,0}^+$ are of pure degree, that is,*

$$H_J^- \cong \mathcal{H}_J^-, \quad H_{J,0}^+ \cong \mathcal{H}_{J,0}^+.$$

By studying the properties of P_J , we get that J is C^∞ pure and full when $\dim \ker P_J = b^2 - 1$.

Theorem 2.5. *Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold, if $\dim \ker P_J = b^2 - 1$, then J is C^∞ pure and full and*

$$\begin{aligned} H_{dR}^2(M; \mathbb{R}) &= H_J^+ \oplus H_J^- \\ &= \text{Span}_{\mathbb{R}}\{\omega\} \oplus H_{J,0}^+ \oplus H_J^- \\ &= H_\omega^{(1,0)}(M; \mathbb{R}) \oplus H_\omega^{(0,2)}(M; \mathbb{R}). \end{aligned}$$

Moreover, J is pure and full.

Recently, L.-S. Tseng and S.-T. Yau [22] introduced new cohomologies for a closed symplectic manifold M . On a compact symplectic manifold (M, ω) of dimension $2n$, the symplectic star operator $*_s$ acts on a differential k -form α by

$$\begin{aligned} \alpha \wedge *_s \alpha' &= (\omega^{-1})^k (\alpha, \alpha') d\text{vol} \\ &= \frac{1}{k!} (\omega^{-1})^{i_1 j_1} \dots (\omega^{-1})^{i_k j_k} \alpha_{i_1 \dots i_k} \alpha'_{j_1 \dots j_k} \frac{\omega^n}{n!} \end{aligned}$$

with repeated indices summed over. Note that $*_s *_s = 1$, which follows from Weil's identity [14, 23]

$$*_s \frac{L^r}{r!} B_k = (-1)^{k(k+1)/2} \frac{L^{n-r-k}}{(n-r-k)!} B_k$$

for any primitive k -form B_k . Also by Weil's identity, for any primitive k -form B_k , we can get

$$*_g \frac{L^r}{r!} B_k = (-1)^{k(k+1)/2} \frac{L^{n-r-k}}{(n-r-k)!} \mathcal{J}(B_k), \quad (1.1)$$

where $\mathcal{J} = \sum_{p,q} (\sqrt{-1})^{p-q} \Pi^{p,q}$ projects a k -form onto its (p, q) parts times the multiplicative factor $(\sqrt{-1})^{p-q}$. The adjoint of the standard exterior derivative takes the form

$$d^\Lambda = (-1)^{k+1} *_s d *_s.$$

By using the properties $d^2 = (d^\Lambda)^2 = 0$ and the anti-commutively $dd^\Lambda = -d^\Lambda d$, Tseng and Yau [22] considered new symplectic cohomology groups $H_{d+d^\Lambda}^k(M)$ and $H_{dd^\Lambda}^k(M)$. They also proved that the space of $d + d^\Lambda$ -harmonic k -forms $\mathcal{H}_{d+d^\Lambda}^k(M)$ and the space of dd^Λ -harmonic k -forms $\mathcal{H}_{dd^\Lambda}^k(M)$ are finite dimensional and isomorphic to $H_{d+d^\Lambda}^k(M)$ and $H_{dd^\Lambda}^k(M)$, respectively.

By considering the relationship between $H_J^- (\cong \mathcal{H}_J^-)$ and symplectic cohomology groups on a closed almost Kähler 4-manifold, we obtain the following theorem.

Theorem 3.2. *Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold, then*

$$\ker P_J = \mathcal{H}_{d+d^\Lambda}^-(M) \cap \mathcal{H}_{dd^\Lambda}^-(M).$$

If $\mathcal{H}_{d+d^\Lambda}^-(M) = \mathcal{H}_{dd^\Lambda}^-(M)$, then

$$\mathcal{H}_{d+d^\Lambda}^2(M) = \mathcal{H}_{dd^\Lambda}^2(M) = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+.$$

In particular, if $n = 2$,

$$\mathcal{H}_{d+d^\Lambda}^-(M) = \mathcal{H}_J^- \oplus \mathcal{H}_g^- \oplus (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)^{-, \perp}_{d+d^\Lambda},$$

$$\mathcal{H}_{dd^\Lambda}^-(M) = \mathcal{H}_J^- \oplus \mathcal{H}_g^- \oplus (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)^{-, \perp}_{dd^\Lambda},$$

$$*_g(\mathcal{H}_J^- \oplus \mathcal{H}_g^-)^{-, \perp}_{d+d^\Lambda} = (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)^{-, \perp}_{dd^\Lambda}.$$

2 Primitive de Rham cohomology of degree two

An almost Kähler structure on a real manifold M of dimension $2n$ is given by a triple (g, J, ω) of a Riemannian metric g , an almost complex structure J and a symplectic form ω , which satisfies the compatibility relation

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot). \quad (2.1)$$

We say that the almost complex structure J is ω compatible if it induces a Riemannian metric via (2.1).

Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold. The almost complex structure J acts on the space Ω^2 of smooth 2-forms on M as an involution by

$$\alpha \longmapsto \alpha(J\cdot, J\cdot), \quad \alpha \in \Omega^2(M). \quad (2.2)$$

This gives the J -invariant, J -anti-invariant decomposition of 2-forms (see [6]):

$$\Omega^2 = \Omega_J^+ \oplus \Omega_J^-, \quad \alpha = \alpha_J^+ + \alpha_J^-$$

as well as the splitting of corresponding vector bundles

$$\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-. \quad (2.3)$$

Let \mathcal{Z}^2 denote the space of closed 2-forms on M and set

$$\mathcal{Z}_J^+ \triangleq \mathcal{Z}^2 \cap \Omega_J^+, \quad \mathcal{Z}_J^- \triangleq \mathcal{Z}^2 \cap \Omega_J^-.$$

Define the J -invariant and J -anti-invariant cohomology subgroups H_J^\pm (see [18]) by

$$H_J^\pm = \{\mathfrak{a} \in H_{dR}^2(M; \mathbb{R}) \mid \text{there exists } \alpha \in \mathcal{Z}_J^\pm \text{ such that } \mathfrak{a} = [\alpha]\}.$$

Let us denote by h_J^+ and h_J^- the dimensions of H_J^+ and H_J^- , respectively. We say J is C^∞ pure if $H_J^+ \cap H_J^- = \{0\}$, C^∞ full if $H_J^+ + H_J^- = H_{dR}^2(M, \mathbb{R})$, and J is C^∞ pure and full if

$$H_{dR}^2(M, \mathbb{R}) = H_J^+ \oplus H_J^-.$$

T. Draghici, T.-J. Li and W. Zhang have proved that for any closed almost complex 4-manifold (M, J) , J is C^∞ pure and full (see [9]).

On a smooth closed manifold M , the space $\Omega^*(M)$ of smooth forms is a vector space, and with C^∞ topology, it is a Fréchet space. The space $\mathcal{E}_*(M)$ of currents is the topological dual space, which is also a Fréchet space (see [13, 19]). As a topological vector space, $\Omega^*(M)$ is reflexive, thus it is also the dual space of $\mathcal{E}_*(M)$. Denote the space of closed currents by $\mathcal{Z}_*(M)$ and the space of boundaries by $\mathcal{B}_*(M)$. On a closed almost complex manifold (M, J) , there is a natural action of J on the space $\Omega^k(M)_\mathbb{C} \triangleq \Omega^k(M) \otimes \mathbb{C}$, which induces a topological type decomposition

$$\Omega^k(M)_\mathbb{C} = \bigoplus_{p+q=k} \Omega_J^{p,q}(M)_\mathbb{C}.$$

If k is even, J also acts on $\Omega^k(M)$ as an involution. Specifically, if $k = 2$, J acts on $\Omega^2(M)$ as (2.2) and decomposes it into the topological direct sum of the invariant part Ω_J^+ and the anti-invariant part Ω_J^- . In this case, the two decompositions are related in the following way:

$$\Omega_J^+(M) = \Omega_J^{1,1}(M)_\mathbb{R} \triangleq \Omega_J^{1,1}(M)_\mathbb{C} \cap \Omega^2(M),$$

$$\Omega_J^-(M) = \Omega_J^{(2,0),(0,2)}(M)_\mathbb{R} \triangleq (\Omega_J^{(2,0)}(M)_\mathbb{C} \oplus \Omega_J^{(0,2)}(M)_\mathbb{C}) \cap \Omega^2(M).$$

For the space of real 2-currents, we have a similar decomposition

$$\mathcal{E}_2(M) = \mathcal{E}_{1,1}^J(M)_\mathbb{R} \oplus \mathcal{E}_{(2,0),(0,2)}^J(M)_\mathbb{R},$$

and the corresponding subspaces of closed and boundary currents,

$$\mathcal{B}_{1,1}^J \subset \mathcal{Z}_{1,1}^J \subset \mathcal{E}_{1,1}^J(M)_\mathbb{R},$$

$$\mathcal{B}_{(2,0),(0,2)}^J \subset \mathcal{Z}_{(2,0),(0,2)}^J \subset \mathcal{E}_{(2,0),(0,2)}^J(M)_\mathbb{R}.$$

We note the dual space of $\mathcal{E}_{1,1}^J(M)_\mathbb{R}$ is $\Omega_J^{1,1}(M)_\mathbb{R}$, and vice versa. Similarly, $\mathcal{E}_{(2,0),(0,2)}^J(M)_\mathbb{R}$ is the dual space of $\Omega_J^{(2,0),(0,2)}(M)_\mathbb{R}$. If $S = (1, 1)$ or $(2, 0), (0, 2)$ (cf. [12, 18]), define

$$H_S^J(M)_\mathbb{R} = \frac{\mathcal{Z}_S^J}{\mathcal{B}_S^J}.$$

J is said to be pure if

$$\frac{\mathcal{Z}_{1,1}^J}{\mathcal{B}_{1,1}^J} \cap \frac{\mathcal{Z}_{(2,0),(0,2)}^J}{\mathcal{B}_{(2,0),(0,2)}^J} = 0.$$

J is said to be full if

$$\frac{\mathcal{Z}_2}{\mathcal{B}_2} = \frac{\mathcal{Z}_{1,1}^J}{\mathcal{B}_{1,1}^J} + \frac{\mathcal{Z}_{(2,0),(0,2)}^J}{\mathcal{B}_{(2,0),(0,2)}^J}.$$

Therefore, an almost complex structure J is pure and full if and only if

$$H_2(M; \mathbb{R}) = H_{1,1}^J(M)_\mathbb{R} \oplus H_{(2,0),(0,2)}^J(M)_\mathbb{R}, \quad (2.4)$$

where $H_2(M; \mathbb{R})$ is the 2-nd de Rham homology group.

In particular, if (M, g, J, ω) is a closed almost Kähler 4-manifold, then the Hodge star operator $*_g$ gives the well-known self-dual, anti-self-dual decomposition of 2-forms:

$$\Omega^2 = \Omega_g^+ \oplus \Omega_g^-$$

as well as the corresponding splitting of the bundles (see [6, 7])

$$\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-. \quad (2.5)$$

The Hodge-de Rham Laplacian commutes with $*_g$, so the decomposition (2.5) holds for the space \mathcal{H}_g^2 of harmonic 2-forms as well. By Hodge theory, this induces cohomology decomposition by the metric g :

$$H_{dR}^2(M; \mathbb{R}) \cong \mathcal{H}_g^2 = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-. \quad (2.6)$$

One defines (see [7])

$$H_g^\pm = \{\mathfrak{a} \in H_{dR}^2(M; \mathbb{R}) \mid \mathfrak{a} = [\alpha] \text{ for some } \alpha \in \mathcal{Z}_g^\pm := \mathcal{Z}^2 \cap \Omega_g^\pm\}.$$

It is easy to see that

$$H_g^\pm \cong \mathcal{Z}_g^\pm = \mathcal{H}_g^\pm$$

and (2.6) can be written as

$$H_{dR}^2(M; \mathbb{R}) = H_g^+ \oplus H_g^-.$$

There are the following relations between the decompositions (2.3) and (2.5) on an almost Kähler 4-manifold (cf. [6, 7]):

$$\begin{aligned} \Lambda_J^+ &= \langle \omega \rangle \oplus \Lambda_g^-, \\ \Lambda_g^+ &= \langle \omega \rangle \oplus \Lambda_J^-, \\ \Lambda_J^+ \cap \Lambda_g^+ &= \langle \omega \rangle, \quad \Lambda_J^- \cap \Lambda_g^- = \{0\}. \end{aligned}$$

It is easy to see that $\mathcal{Z}_J^- \subset \mathcal{H}_g^+$ and $H_g^- \subset H_J^+$. Let b^2 , b^+ and b^- be the second, the self-dual and the anti-self-dual Betti number of M , respectively. Thus $b^2 = b^+ + b^-$. It is easy to see that, for a closed almost Kähler 4-manifold (M, g, J, ω) , there hold (see [9, 10, 21]):

$$H_J^- \cong \mathcal{Z}_J^-, \quad h_J^+ + h_J^- = b^2, \quad h_J^+ \geq b^- + 1, \quad 0 \leq h_J^- \leq b^+ - 1. \quad (2.7)$$

Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold. A differential k -form B_k with $k \leq n$ is called primitive if $L^{n-k+1}B_k = 0$ (see [22, 24]). Here L is the Lefschetz operator (see [4, 22, 24]) which is defined acting on a k -form $A_k \in \Omega^k(M)$ by

$$L(A_k) = \omega \wedge A_k.$$

Define the space of primitive k -forms by Ω_0^k . Specifically,

$$\Omega_0^2 = \{\alpha \in \Omega^2 : \omega^{n-1} \wedge \alpha = 0\}.$$

Therefore,

$$\Omega^2 = \Omega_1^2 \oplus \Omega_0^2,$$

where $\Omega_1^2 \triangleq \{f\omega : f \in C^\infty(M)\}$. It is easy to see that $\Omega_J^- \subset \Omega_0^2$. So we can get the following decomposition

$$\Omega_0^2 = \Omega_{J,0}^+ \oplus \Omega_J^-, \quad (2.8)$$

where $\Omega_{J,0}^+$ is the space of the primitive J -invariant 2-forms. We consider the following second order linear differential operator on Ω_0^2 .

$$\begin{aligned} P_J : \Omega_0^2 &\rightarrow \Omega_0^2 \\ \psi &\mapsto \Delta_g \psi - \frac{1}{n} g(\Delta_g \psi, \omega) \omega, \end{aligned}$$

where Δ_g is the Riemannian Laplacian with respect to the metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ (here we use the convention $g(\omega, \omega) = n$).

Lemma 2.1. *P_J is a self-adjoint strongly elliptic linear operator with kernel the primitive g -harmonic 2-forms.*

Proof. We claim that

$$\begin{aligned} (P_J - \Delta_g)(\psi) &= -\frac{1}{n} [d^*(\nabla^g \psi, \omega)_g - (\nabla^g \psi, \nabla^g \omega)_g - (2 - \frac{4}{n-2}) \text{Tr}(\omega \cdot \text{Ric} \cdot \psi) \\ &\quad - \frac{2s^g}{(n-1)(n-2)} (\psi, \omega)_g - W^g(\psi, \omega)] \omega \\ &= -\frac{1}{n} [-d^*(\psi, \nabla^g \omega)_g - (\nabla^g \psi, \nabla^g \omega)_g - (2 - \frac{4}{n-2}) \text{Tr}(\omega \cdot \text{Ric} \cdot \psi) \\ &\quad - \frac{2s^g}{(n-1)(n-2)} (\psi, \omega)_g - W^g(\psi, \omega)] \omega, \end{aligned}$$

where W^g is the Weyl tensor (see [3]), ∇^g is the Levi-Civita connection, s^g is the Riemannian scalar curvature with respect to the metric g and $\text{Tr}(\omega \cdot \text{Ric} \cdot \psi) = \Sigma_{i,j,k} \omega_{ij} R_{jk} \psi_{ik}$ is the trace of $\omega \otimes \text{Ric} \otimes \psi$. Indeed, by Weitzenböck-Bochner formula (see [3]), we will get

$$\begin{aligned} (\Delta_g \psi - \text{Tr}(\nabla^g)^2 \psi, \omega)_g &= (\Delta_g \psi - d^*(\nabla^g) \psi, \omega)_g \\ &= \sum_{i,j,k,l} 2R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,j,k,l} R_{ik} \psi_{kj} \omega_{ij} - \sum_{i,j,k,l} R_{jk} \psi_{ik} \omega_{ij} \\ &= \sum_{i,j,k,l} 2R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,j,k,l} R_{ik} \psi_{kj} \omega_{ij} - \sum_{i,j,k,l} R_{ik} \psi_{jk} \omega_{ji} \\ &= \sum_{i,j,k,l} 2R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,j,k,l} 2R_{ik} \psi_{kj} \omega_{ij}. \end{aligned} \quad (2.9)$$

On the other hand,

$$\begin{aligned}
W^g(\omega, \psi) &= (R - \frac{1}{n-2} Ric \circ g + \frac{s^g}{2(n-1)(n-2)} g \circ g)(\omega, \psi) \\
&= \sum_{i,j,k,l} \omega_{ij} \psi_{kl} R_{ijkl} - \sum_{i,j,k,l} \frac{1}{n-2} \omega_{ij} \psi_{kl} (R_{il} g_{jk} + R_{jk} g_{il} \\
&\quad - R_{ik} g_{jl} - R_{jl} g_{ik}) + \frac{s^g}{2(n-1)(n-2)} g \circ g(\omega, \psi) \\
&= \sum_{i,j,k,l} \omega_{ij} \psi_{kl} R_{ijkl} - \sum_{i,k,l} \frac{4}{n-2} \omega_{ik} \psi_{kl} R_{il} - \sum_{i,j} \frac{2s^g}{(n-1)(n-2)} \omega_{ij} \psi_{ij} \\
&= \sum_{i,j,k,l} \omega_{ij} \psi_{kl} (-R_{iljk} - R_{iklj}) - \sum_{i,k,l} \frac{4}{n-2} \omega_{ik} \psi_{kl} R_{il} \\
&\quad - \sum_{i,j} \frac{2s^g}{2(n-1)(n-2)} \omega_{ij} \psi_{ij} \\
&= \sum_{i,j,k,l} -2R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,k,l} \frac{4}{n-2} R_{il} \psi_{kl} \omega_{ik} \\
&\quad - \sum_{i,j} \frac{2s^g}{2(n-1)(n-2)} \omega_{ij} \psi_{ij}. \tag{2.10}
\end{aligned}$$

Here we compute (2.9) and (2.10) under the local coordinates system $(x^1, x^2, \dots, x^{2n})$. In addition, we suppose (ω_{ij}) to be the local representation of ω . Similarly, $\psi = (\psi_{ij})$, $Ric = (R_{ij})$ and $R = (R_{ijkl})$, where Ric is the Ricci curvature tensor and R is the Riemannian curvature tensor with respect to metric g . By (2.9) and (2.10), we can get the above claim. It is easy to see that $P_J - \Delta_g$ is a linear differential operator of order 1. So the operator P_J is a self-adjoint strongly elliptic linear operator of order 2.

It remains to prove that the kernel of P_J is the space of the primitive g -harmonic 2-forms, that is, $\mathcal{H}_g^2 \cap \Omega_0^2$. Clearly, $\mathcal{H}_g^2 \cap \Omega_0^2 \subset \ker P_J$. For any $\psi \in \ker P_J$,

$$\begin{aligned}
0 &= \int_M (P_J(\psi), \psi)_g dvol_g \\
&= \int_M (\Delta_g \psi - \frac{1}{n} (\Delta_g \psi, \omega)_g \omega, \psi)_g \\
&= \int_M (\Delta_g \psi, \psi)_g \\
&= \int_M (d\psi, d\psi)_g + (d^* \psi, d^* \psi)_g.
\end{aligned}$$

Hence, $d\psi = d^* \psi = 0$ and ψ is a primitive g -harmonic 2-form. So we get $\ker P_J = \mathcal{H}_g^2 \cap \Omega_0^2$. \square

Remark 2.2. Let J_t , $t \in [0, 1]$ be a smooth family of ω -compatible almost complex structures on M , then $\dim \ker P_{J_t}$ is an upper-semi-continuous function in t , by a classical result of Kodaira and Morrow showing the upper-semi-continuity of the kernel of a family of elliptic differential operators (Theorem 4.3 in [15]).

We define \mathcal{H}_J^- to be the space of the harmonic J -anti-invariant 2-forms and $\mathcal{H}_{J,0}^+$ to be the space of the harmonic primitive J -invariant 2-forms. Define the primitive

J -invariant cohomology subgroup $H_{J,0}^+$ by

$$H_{J,0}^+ = \{\mathfrak{a} \in H_{dR}^2(M; \mathbb{R}) \mid \text{there exists } \alpha \in \mathcal{Z}^2 \cap \Omega_{J,0}^+ \text{ such that } \mathfrak{a} = [\alpha]\}.$$

Proposition 2.3. *Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold, then $\ker P_J = \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+$ and the harmonic representatives of H_J^- and $H_{J,0}^+$ are of pure degree, that is,*

$$H_J^- \cong \mathcal{H}_J^-, \quad H_{J,0}^+ \cong \mathcal{H}_{J,0}^+.$$

Proof. For any $\alpha \in \ker P_J$, α is primitive and harmonic. So $d\alpha = 0, d *_g \alpha = 0$ and α can be written as $\alpha = \beta_\alpha + \gamma_\alpha$, where $\beta_\alpha \in \Omega_J^-$ and $\gamma_\alpha \in \Omega_{J,0}^+$. By a direct computation, we get

$$*_g \alpha = \frac{L^{n-2}}{(n-2)!}(\beta_\alpha - \gamma_\alpha)$$

and

$$\left(\frac{L^{n-2}}{(n-2)!} + *_g\right)\alpha = 2\frac{L^{n-2}}{(n-2)!}\beta_\alpha.$$

Hence,

$$d\left(2\frac{L^{n-2}}{(n-2)!}\beta_\alpha\right) = d\left(\frac{L^{n-2}}{(n-2)!} + *_g\right)\alpha = 0.$$

So we can get $d\beta_\alpha = 0$. Since $*_g \beta_\alpha = \frac{L^{n-2}}{(n-2)!}\beta_\alpha$, $d *_g \beta_\alpha = 0$. Therefore, $\beta_\alpha \in \mathcal{H}_J^-$. Similarly,

$$\left(\frac{L^{n-2}}{(n-2)!} - *_g\right)\alpha = 2\frac{L^{n-2}}{(n-2)!}\gamma_\alpha$$

and we can get $\gamma_\alpha \in \mathcal{H}_{J,0}^+$. Thus, $\ker P_J = \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+$.

For any $\mathfrak{a} = [\alpha] \in H_J^-, \alpha \in \mathcal{Z}_J^-$. By (1.1),

$$*_g \alpha = -\frac{L^{n-2}}{(n-2)!}\mathcal{J}(\alpha) = \frac{L^{n-2}}{(n-2)!}\alpha.$$

So $d *_g \alpha = d\frac{L^{n-2}}{(n-2)!}\alpha = 0$ and α is a harmonic J -anti-invariant form, that is, $\alpha \in \mathcal{H}_J^-$. Hence, $H_J^- \cong \mathcal{H}_J^-$. Similarly, $H_{J,0}^+ \cong \mathcal{H}_{J,0}^+$. The harmonic representatives of H_J^- and $H_{J,0}^+$ are of pure degree. \square

Remark 2.4. In case $n = 2$, on a closed almost Kähler 4-manifold, Lejmi [16] proved that P_J preserves the decomposition

$$\Omega_0^2 = \Omega_{J,0}^+ \oplus \Omega_J^-.$$

Furthermore, $P_J|_{\Omega_{J,0}^+}(\psi) = \Delta_g \psi$ and $P_J|_{\Omega_J^-}(\psi) = 2d_J^- d^* \psi$. He also pointed out that $P_J|_{\Omega_J^-}(\psi) = 2d_J^- d^* \psi$ is a self-adjoint strongly elliptic linear operator from Ω_J^- to Ω_J^- on a closed almost Kähler 4-manifold. It follows that the kernel of P_J consists of primitive harmonic 2-forms which splits as anti-self-dual and J -anti-invariant ones. So he gets

$$\dim \ker P_J = b^- + h_J^-.$$

But when $n > 2$, by computing the principal symbol of $d_J^- d^*$, one finds that $d_J^- d^*$ is no longer a self-adjoint strongly elliptic linear operator. So we are not able to get any good properties about the $\dim \ker P_J$ in higher dimension.

In [1], D. Angella and A. Tomassini define

$$H_{\omega}^{(r,s)}(M; \mathbb{R}) \triangleq \{[L^r \beta] \in H_{dR}^{2r+s}(M; \mathbb{R}) : \beta \in \Omega_0^s\} \subseteq H_{dR}^{2r+s}(M; \mathbb{R})$$

for $r, s \in \mathbb{N}$. Obviously, for every $k \in \mathbb{N}$, one has

$$\sum_{2r+s=k} H_{\omega}^{(r,s)}(M; \mathbb{R}) \subseteq H_{dR}^{2r+s}(M; \mathbb{R}).$$

A natural question is that when the above inclusion is actually an equality, and when the sum is a direct sum. Fortunately, Angella and Tomassini have proved that

$$H_{dR}^2(M; \mathbb{R}) = H_{\omega}^{(1,0)}(M; \mathbb{R}) \oplus H_{\omega}^{(0,2)}(M; \mathbb{R}) \quad (2.11)$$

in [1]. Clearly, $H_{\omega}^{(1,0)}(M; \mathbb{R}) \cong \text{Span}_{\mathbb{R}}\{\omega\}$ and $\dim H_{\omega}^{(0,2)}(M; \mathbb{R}) = b^2 - 1$. Here we want to emphasize that $\ker P_J \cong H_J^- \oplus H_{J,0}^+ \subseteq H_{\omega}^{(0,2)}(M; \mathbb{R})$ and if $\dim \ker P_J = b^2 - 1$, then $\ker P_J \cong H_J^- \oplus H_{J,0}^+ = H_{\omega}^{(0,2)}(M; \mathbb{R})$.

It is well known that on any closed almost complex 4-manifold (M, J) , J is C^∞ pure and full (see [9]). But we can not get this result in higher dimension. In [12], A. Fino and A. Tomassini showed the existence of a compact 6-dimensional nil-manifold with an almost complex structure which is not C^∞ pure, i.e., the intersection of $H_J^+(M)$ and $H_J^-(M)$ is non-empty. They also prove that on a compact almost complex $2n$ -manifold (M, J) , if J admits a compatible symplectic structure, then J is C^∞ pure. With this result and by studying the dimension of $\dim \ker P_J$, we can get the following theorem.

Theorem 2.5. *Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold, if $\dim \ker P_J = b^2 - 1$, then J is C^∞ pure and full and*

$$\begin{aligned} H_{dR}^2(M; \mathbb{R}) &= H_J^+ \oplus H_J^- \\ &= \text{Span}_{\mathbb{R}}\{\omega\} \oplus H_{J,0}^+ \oplus H_J^- \\ &= H_{\omega}^{(1,0)}(M; \mathbb{R}) \oplus H_{\omega}^{(0,2)}(M; \mathbb{R}). \end{aligned}$$

Moreover, J is pure and full.

Proof. In [12], A. Fino and A. Tomassini showed that J is C^∞ pure if (M, g, J, ω) is a closed almost Kähler $2n$ -manifold. In [9], T. Draghici, T.-J. Li and W. Zhang proved the same result on a closed almost Kähler $2n$ -manifold. Next, we will prove that J is C^∞ full under the condition of $\dim \ker P_J = b^2 - 1$.

By proposition 2.3, if $\dim \ker P_J = b^2 - 1$, we get

$$\dim \mathcal{H}_J^- + \dim \mathcal{H}_{J,0}^+ = b^2 - 1.$$

Hence,

$$b^2 = \dim \mathcal{H}_J^- + (\dim \mathcal{H}_{J,0}^+ + 1). \quad (2.12)$$

It is easy to see that

$$H_J^+ \oplus H_J^- \subseteq H_{dR}^2(M; \mathbb{R}),$$

and

$$\text{Span}_{\mathbb{R}}\{\omega\} \oplus H_{J,0}^+ \subseteq H_J^+.$$

So we will get

$$h_J^+ + h_J^- \leq b^2 \quad (2.13)$$

and

$$\dim H_{J,0}^+ + 1 = \dim \mathcal{H}_{J,0}^+ + 1 \leq h_J^+. \quad (2.14)$$

Therefore, by (2.12), (2.13) and (2.14),

$$h_J^+ + h_J^- \leq b^2 = \dim \mathcal{H}_J^- + (\dim \mathcal{H}_{J,0}^+ + 1) \leq h_J^+ + h_J^-.$$

Clearly, $h_J^+ + h_J^- = b^2$ and $\dim \mathcal{H}_{J,0}^+ + 1 = h_J^+$. So we can get

$$H_{dR}^2(M; \mathbb{R}) = H_J^+ \oplus H_J^-$$

and

$$\text{Span}_{\mathbb{R}}\{\omega\} \oplus H_{J,0}^+ = H_J^+.$$

Then we can get the following decompositions

$$\begin{aligned} H_{dR}^2(M; \mathbb{R}) &= H_J^+ \oplus H_J^- \\ &= \text{Span}_{\mathbb{R}}\{\omega\} \oplus H_{J,0}^+ \oplus H_J^- \\ &= H_{\omega}^{(1,0)}(M; \mathbb{R}) \oplus H_{\omega}^{(0,2)}(M; \mathbb{R}) \end{aligned}$$

and $H_J^+ \cong \mathcal{H}_J^+$, where \mathcal{H}_J^+ is the space of the harmonic J-invariant 2-forms. Of course, J is C^∞ pure and full.

We have proven that J is C^∞ pure and full. Then by Theorem 3.7 in [12], we can get that J is pure. In addition, in the above proof, we have gotten that $H_J^- \cong \mathcal{H}_J^-$ and $H_J^+ \cong \mathcal{H}_J^+$ when $\dim \ker P_J = b^2 - 1$. Hence the harmonic representatives of $H_{dR}^2(M; \mathbb{R})$ are of pure degree (cf. [2]). Also by Theorem 3.7 in [12], we can get that J is pure and full. This completes the proof of the Theorem. \square

In the above theorem, $\dim \ker P_J = b^2 - 1$ is just the sufficient condition but not the necessary condition for J 's C^∞ pureness and fullness. Just by J 's C^∞ pureness and fullness, we can not get $\dim \ker P_J = b^2 - 1$. Indeed, we have the following counter-example which is constructed by T. Draghici ([8]).

Example 2.6. (also cf. [2]) Let \mathbb{T}^4 be the standard torus with coordinates $\{x^1, x^2, x^3, x^4\}$. Denote by (g_0, J_0, ω_0) be a standard flat Kähler structure on \mathbb{T}^4 , so (\mathbb{T}^4, ω_0) has the hard Lefschetz property (cf. [24]), that is, the map

$$H_{dR}^k(\mathbb{T}^4; \mathbb{R}) \rightarrow H_{dR}^{4-k}(\mathbb{T}^4; \mathbb{R}), \quad \alpha \mapsto [\omega_0]^{2-k} \wedge \alpha,$$

is an isomorphism for all $k \leq 2$. We choose

$$g_0 = \sum_i dx^i \otimes dx^i, \quad \omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4,$$

so J_0 is given by

$$J_0 dx^1 = dx^2, \quad J_0 dx^2 = -dx^1, \quad J_0 dx^3 = dx^4, \quad J_0 dx^4 = -dx^3.$$

Equivalently, J_0 may be given by specifying

$$\Lambda_{J_0}^- = \text{Span}\{dx^1 \wedge dx^3 - dx^2 \wedge dx^4, dx^1 \wedge dx^4 + dx^2 \wedge dx^3\}.$$

Consider the almost complex structure J given by

$$Jdx^1 = m dx^2, \quad Jdx^2 = -\frac{1}{m} dx^1, \quad Jdx^3 = dx^4, \quad Jdx^4 = -dx^3,$$

where $m = m(x^2, x^4)$ is a positive periodic function on x^2, x^4 only. It is easy to see that

$$\Lambda_J^- = \text{Span}\{dx^1 \wedge dx^3 - m dx^2 \wedge dx^4, dx^1 \wedge dx^4 + m dx^2 \wedge dx^3\},$$

and J is compatible with ω_0 .

$$g(\cdot, \cdot) = \omega_0(\cdot, J\cdot) = \frac{1}{m} dx^1 \otimes dx^1 + m dx^2 \otimes dx^2 + dx^3 \otimes dx^3 + dx^4 \otimes dx^4.$$

We claim that we can choose m such that $h_J^- = 1$ (proved by T. Draghici in [8]).

Denote by $\psi_1 = dx^1 \wedge dx^3 - m dx^2 \wedge dx^4$ and $\psi_2 = dx^1 \wedge dx^4 + m dx^2 \wedge dx^3$. Note that $m = m(x^2, x^4)$, we have

$$d\psi_1 = 0, \quad d\psi_2 = m_4 dx^2 \wedge dx^3 \wedge dx^4 = (\log m)_4 dx^4 \wedge \psi_2. \quad (2.15)$$

Here we denote by $m_4 = (\partial m)/(\partial x^4)$ and $(\log m)_4 = (\partial(\log m))/(\partial x^4)$. The general J -anti-invariant form is written as $A\psi_1 + B\psi_2$, where A, B are smooth functions on the torus. The condition that such a form to be closed is equivalent with

$$dA \wedge \psi_1 + (dB + B(\log m)_4 dx^4) \wedge \psi_2 = 0. \quad (2.16)$$

We claim that in any solution (A, B) of (2.16) A must be a constant. To see this, taking the Hodge operator $*_g$ of both sides of the above equation we get

$$JdA = -(dB + B(\log m)_4 dx^4). \quad (2.17)$$

Taking one more differential, and then taking trace with respect to ω_0 we get

$$\Delta A = B_3(\log m)_4, \quad (2.18)$$

where $B_3 = (\partial B)/(\partial x^3)$. By (2.17), we get $B_3 = A_4 := (\partial A)/(\partial x^4)$. Then (2.18) evolves into

$$-\Delta A + A_4(\log m)_4 = 0. \quad (2.19)$$

By the maximum principle, it follows that $A = \text{const}$. Plugging this back in (2.17), we see that B must satisfy

$$dB = -B(\log m)_4 dx^4.$$

It is easy to see that the compatibility relation of this system is

$$B(\log m)_{42} = 0.$$

Thus, if $(\log m)_{42} \neq 0$ (Indeed, it will be sufficient that the locus of $(\log m)_{42}$ has zero Lebesgue measure.), the only solutions for (2.16) are $A = \text{const}$ and $B = 0$, hence $h_J^- = 1$. We choose $m = e^{\sin 2\pi(x^2 + x^4)}$ to be a positive periodic function on \mathbb{T}^4 such that $(\log m)_{42} = -4\pi^2 \sin 2\pi(x^2 + x^4)$ has zero Lebesgue measure. So on $(\mathbb{T}^4, g, J, \omega_0)$, h_J^- is equal to 1. Of course, J is C^∞ pure and full (T. Draghici, T.-J. Li and W. Zhang proved in [9] that on any closed almost complex 4-manifold (M, J) ,

J is C^∞ pure and full.). Additionally, Lejmi [16] proved that, on a closed almost Kähler 4-manifold (M, g, J, ω) , the kernel of P_J consists of primitive harmonic 2-forms and $\dim \ker P_J = b^- + h_J^-$. So on $(\mathbb{T}^4, g, J, \omega_0)$, $\dim \ker P_J = b^- + h_J^- = 4 < 5 = b^2 - 1$.

Let us denote by $e_i \triangleq dx^i$ and $e_{ij} \triangleq dx^i \wedge dx^j$. Please see the following table.

$\mathcal{Z}_{J_0}^+$	$\omega_0, \quad e_{12} - e_{34}, \quad e_{13} + e_{24}, \quad e_{14} - e_{23}$
$\mathcal{Z}_{J_0}^-$	$e_{13} - e_{24}, \quad e_{14} + e_{23}$
$\ker P_{J_0}$	$e_{12} - e_{34}, \quad e_{13} + e_{24}, \quad e_{14} - e_{23}, \quad e_{13} - e_{24}, \quad e_{14} + e_{23}$
$\mathcal{H}_{g_0}^+$	$\omega_0, \quad e_{13} - e_{24}, \quad e_{14} + e_{23}$
$\mathcal{H}_{g_0}^-$	$e_{12} - e_{34}, \quad e_{13} + e_{24}, \quad e_{14} - e_{23}$
\mathcal{Z}_J^+	$\omega_0, \quad e_{12} - e_{34}, \quad e_{13} + me_{24}, \quad \frac{1}{1+m}(e_{12} - e_{34} + e_{14} - me_{23}),$ $\frac{1}{1-m}(e_{12} + e_{34} + e_{14} - me_{23})$
\mathcal{Z}_J^-	$e_{13} - me_{24}$
$\ker P_J$	$e_{13} - me_{24}, \quad e_{12} - e_{34}, \quad e_{13} + me_{24}, \quad \frac{1}{1+m}(e_{12} - e_{34} + e_{14} - me_{23})$
\mathcal{H}_g^+	$\omega_0, \quad e_{13} - me_{24}, \quad \frac{1}{1+m}(\omega_0 + e_{14} + me_{23})$
\mathcal{H}_g^-	$e_{12} - e_{34}, \quad e_{13} + me_{24}, \quad \frac{1}{1+m}(e_{12} - e_{34} + e_{14} - me_{23})$

Table 1. Bases for $\mathcal{Z}_{J_0}^+$, $\mathcal{Z}_{J_0}^-$, $\ker P_{J_0}$, etc. of \mathbb{T}^4 .

By the above example, we want to propose the following question:

Question 2.7. *On any closed symplectic 4-manifold (M, ω) , is there a ω -compatible (or ω -tame) almost complex structure J such that $\dim \ker P_J = b^2 - 1$?*

Remark 2.8. *On any closed symplectic 4-manifold (M, ω) , if an almost complex structure J is tamed by symplectic form ω and $h_J^- = b^+ - 1$, then it implies that there exists the generalized $\partial\bar{\partial}$ -lemma (cf. [17, 20]).*

The above example \mathbb{T}^4 admits a Kähler structure (g_0, J_0, ω_0) . In Section 3, we will give a non-Kähler example $M^6(c)$ which is constructed by M. Fernández, V. Muñoz and J. A. Santisteban. They have proven that $M^6(c)$ does not admit any Kähler metric (cf. [11]). We will prove that there exists an almost complex structure J on $M^6(c)$ such that $\dim \ker P_J = b^2 - 1$. Please see the Example 3.3.

3 Primitive symplectic cohomology of degree two

L.-S. Tseng and S.-T. Yau [22] considered new symplectic cohomology groups

$$H_{d+d^\Lambda}^k(M) = \frac{\text{Ker}(d + d^\Lambda) \cap \Omega^k(M)}{\text{Im}(dd^\Lambda) \cap \Omega^k(M)}.$$

and

$$H_{dd^\Lambda}^k(M) = \frac{\text{Ker}(dd^\Lambda) \cap \Omega^k(M)}{(\text{Im } d + \text{Im } d^\Lambda) \cap \Omega^k(M)}$$

on a compact symplectic manifold (M, ω) of dimension $2n$. We denote the spaces of $d + d^\Lambda$ harmonic k -forms and dd^Λ harmonic k -forms by $\mathcal{H}_{d+d^\Lambda}^k(M)$ and $\mathcal{H}_{dd^\Lambda}^k(M)$,

respectively. For any almost Kähler triple (g, J, ω) , a k -form $\alpha \in \Omega^k(M)$ is said to be $d + d^\Lambda$ -harmonic (see [22]) if

$$d\alpha = d^\Lambda \alpha = 0 \quad \text{and} \quad (dd^\Lambda)^* \alpha = 0,$$

and dd^Λ -harmonic (see [22]) if

$$d^* \alpha = (d^\Lambda)^* \alpha = 0 \quad \text{and} \quad dd^\Lambda \alpha = 0,$$

where $d^* = -*_g d *_g$, $d^{\Lambda*} = *_g d^\Lambda *_g$ and $(dd^\Lambda)^* = (-1)^{k+1} *_g dd^\Lambda *_g$. Tseng and Yau also proved that $\mathcal{H}_{d+d^\Lambda}^k(M)$ and $\mathcal{H}_{dd^\Lambda}^k(M)$ are finite dimensional and isomorphic to $H_{d+d^\Lambda}^k(M)$ and $H_{dd^\Lambda}^k(M)$, respectively. Let $\mathcal{H}_{d+d^\Lambda}^-(M)$ and $\mathcal{H}_{dd^\Lambda}^-(M)$ denote the spaces of primitive $d + d^\Lambda$ harmonic 2-forms and primitive dd^Λ harmonic 2-forms, respectively.

$$\mathcal{H}_{d+d^\Lambda}^-(M) \triangleq \mathcal{H}_{d+d^\Lambda}^2(M) \cap \Omega_0^2, \quad \mathcal{H}_{dd^\Lambda}^-(M) \triangleq \mathcal{H}_{dd^\Lambda}^2(M) \cap \Omega_0^2.$$

Hence, we can get the following decompositions

$$\mathcal{H}_{d+d^\Lambda}^2(M) = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_{d+d^\Lambda}^-(M),$$

$$\mathcal{H}_{dd^\Lambda}^2(M) = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_{dd^\Lambda}^-(M).$$

Definition 3.1. Let (M, g, J, ω) be a closed almost Kähler 4-manifold. Set

$$(\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{d+d^\Lambda}^{-,\perp} = \{\alpha \in \mathcal{H}_{d+d^\Lambda}^- \mid \alpha = d_J^- \theta^1 + d_g^- \theta^2\}$$

and

$$(\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{dd^\Lambda}^{-,\perp} = \{\alpha \in \mathcal{H}_{dd^\Lambda}^- \mid \alpha = d_J^- \theta^1 + d_g^- \theta^2\},$$

where $d_J^- = P_J^- \circ d$, $d_g^- = P_g^- \circ d$ and $\theta^1, \theta^2 \in \Omega^1(M)$. Here P_J^- is the projection from $\Omega^2(M)$ to $\Omega_J^-(M)$ and P_g^- is the projection from $\Omega^2(M)$ to $\Omega_g^-(M)$.

Theorem 3.2. Suppose that (M, g, J, ω) is a closed almost Kähler $2n$ -manifold, then

$$\ker P_J = \mathcal{H}_{d+d^\Lambda}^-(M) \cap \mathcal{H}_{dd^\Lambda}^-(M).$$

If $\mathcal{H}_{d+d^\Lambda}^-(M) = \mathcal{H}_{dd^\Lambda}^-(M)$, then

$$\mathcal{H}_{d+d^\Lambda}^2(M) = \mathcal{H}_{dd^\Lambda}^2(M) = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+.$$

In particular, if $n = 2$,

$$\mathcal{H}_{d+d^\Lambda}^-(M) = \mathcal{H}_J^- \oplus \mathcal{H}_g^- \oplus (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{d+d^\Lambda}^{-,\perp},$$

$$\mathcal{H}_{dd^\Lambda}^-(M) = \mathcal{H}_J^- \oplus \mathcal{H}_g^- \oplus (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{dd^\Lambda}^{-,\perp},$$

$$*_g(\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{d+d^\Lambda}^{-,\perp} = (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{dd^\Lambda}^{-,\perp}.$$

Proof. Let us begin with the first assertion of the Theorem. We claim that $\mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{d+d^\Lambda}^-(M)$ and $\mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{dd^\Lambda}^-(M)$. Indeed, for any $\alpha = \beta + \gamma \in \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+$, $\beta \in \mathcal{H}_J^-$, $\gamma \in \mathcal{H}_{J,0}^+$. By Weil's identity, we have $*_s \alpha = -\frac{L^{n-2}}{(n-2)!} \alpha$. Then

$$d *_s \alpha = -d \frac{L^{n-2}}{(n-2)!} \alpha = -\frac{L^{n-2}}{(n-2)!} d\alpha = 0.$$

Hence, $d^\Lambda \alpha = 0$. Also by Weil's identity, we have

$$*_s *_g \alpha = *_s *_g \beta + *_s *_g \gamma = *_s \frac{L^{n-2}}{(n-2)!} \beta - *_s \frac{L^{n-2}}{(n-2)!} \gamma = -\beta + \gamma.$$

Hence, $d *_s *_g \alpha = d(-\beta + \gamma) = 0$. So we get $(dd^\Lambda)^* \alpha = 0$. Therefore, $\alpha \in \mathcal{H}_{d+d^\Lambda}^-(M)$ and $\mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{d+d^\Lambda}^-(M)$. It is similar for $\mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{dd^\Lambda}^-(M)$. In particular, if $n = 2$, $\mathcal{H}_{J,0}^+ = \mathcal{H}_g^-$, we can get $\mathcal{H}_J^- \oplus \mathcal{H}_g^- \subset \mathcal{H}_{d+d^\Lambda}^-(M)$ and $\mathcal{H}_J^- \oplus \mathcal{H}_g^- \subset \mathcal{H}_{dd^\Lambda}^-(M)$. So we can get

$$\mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{d+d^\Lambda}^-(M) \cap \mathcal{H}_{dd^\Lambda}^-(M).$$

For the other hand, it follows straightforwardly from the definitions of $\mathcal{H}_{d+d^\Lambda}^-(M)$, $\mathcal{H}_{dd^\Lambda}^-(M)$ and $\ker P_J$. If $\mathcal{H}_{d+d^\Lambda}^-(M) = \mathcal{H}_{dd^\Lambda}^-(M)$ (i.e. $\mathcal{H}_{d+d^\Lambda}^2(M) = \mathcal{H}_{dd^\Lambda}^2(M)$), then $\ker P_J = \mathcal{H}_{d+d^\Lambda}^-(M) = \mathcal{H}_{dd^\Lambda}^-(M)$. Hence,

$$\mathcal{H}_{d+d^\Lambda}^2(M) = \mathcal{H}_{dd^\Lambda}^2(M) = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \ker P_J = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+.$$

In the following, we suppose that (M, g, J, ω) is a closed almost Kähler 4-manifold. Then it is easy to see that $\Omega_g^- = \Omega_{J,0}^+$. So Ω^2 can be written as

$$\Omega^2 = \Omega_1^2 \oplus \Omega_0^2 = \Omega_1^2 \oplus \Omega_g^- \oplus \Omega_J^-.$$

For any $\alpha \in \mathcal{H}_{d+d^\Lambda}^2$, by the definition of $\mathcal{H}_{d+d^\Lambda}^2$,

$$d\alpha = d^\Lambda \alpha = 0, \quad (dd^\Lambda)^* \alpha = 0,$$

where $(dd^\Lambda)^* = - *_g dd^\Lambda *_g$. It is clear that

$$d\alpha = 0, \quad d *_s \alpha = 0.$$

Hence

$$d \frac{1}{2} (1 + *_s) \alpha = 0.$$

Since $\frac{1}{2} (1 + *_s) \alpha \in \Omega_1^2$, it can be written as

$$\frac{1}{2} (1 + *_s) \alpha = f_\alpha \omega.$$

Since $d\omega = 0$, we have $d(f_\alpha \omega) = df_\alpha \wedge \omega = 0$. It follows that $f_\alpha = c_\alpha$ is a constant since ω is nondegenerate.

Let

$$\alpha_1 = \alpha - c_\alpha \omega = \frac{1}{2} (1 - *_s) \alpha \in \Omega_0^2 = \Omega_J^- \oplus \Omega_g^-.$$

Hence, α_1 is still in $\mathcal{H}_{d+d^\Lambda}^2$ and α_1 can be written as

$$\alpha_1 = \alpha_{1,J}^- + \alpha_{1,g}^-,$$

where $\alpha_{1,J}^- \in \Omega_J^-$ and $\alpha_{1,g}^- \in \Omega_g^-$. By Lejmi lemma [16] and Hodge decomposition [7],

$$\alpha_{1,J}^- = \beta_\alpha + d_J^- d^* \eta_\alpha = \beta_\alpha + d_J^- \theta_\alpha^1, \quad \theta_\alpha^1 = d^* \eta_\alpha$$

and

$$\alpha_{1,g}^- = \gamma_\alpha + d_g^- d^* \xi_\alpha = \gamma_\alpha + d_g^- \theta_\alpha^2, \quad \theta_\alpha^2 = d^* \xi_\alpha,$$

where $\beta_\alpha \in \mathcal{Z}_J^- = \mathcal{H}_J^-$, $\gamma_\alpha \in \mathcal{H}_g^-$, $\eta_\alpha \in \Omega_J^-$ and $\xi_\alpha \in \Omega_g^-$. Then

$$\alpha = c_\alpha \omega + \alpha_1 = c_\alpha \omega + \beta_\alpha + \gamma_\alpha + (d_J^- \theta_\alpha^1 + d_g^- \theta_\alpha^2)$$

and $(d_J^- \theta_\alpha^1 + d_g^- \theta_\alpha^2) \in (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{d+d^\Lambda}^{-,\perp}$. So we can get that

$$\mathcal{H}_{d+d^\Lambda}^2 = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_J^- \oplus \mathcal{H}_g^- \oplus (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{d+d^\Lambda}^{-,\perp}.$$

It is easy to see that

$$(\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{d+d^\Lambda}^{-,\perp} \subset \mathcal{H}_{d+d^\Lambda}^-$$

is just the orthogonal complement of $\mathcal{H}_J^- \oplus \mathcal{H}_g^-$ in $\mathcal{H}_{d+d^\Lambda}^-$ with respect to the cup product. Similarly, one has

$$\mathcal{H}_{dd^\Lambda}^2 = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_J^- \oplus \mathcal{H}_g^- \oplus (\mathcal{H}_J^- \oplus \mathcal{H}_g^-)_{dd^\Lambda}^{-,\perp},$$

since $*_g \mathcal{H}_{d+d^\Lambda}^2 = \mathcal{H}_{dd^\Lambda}^2$ and $*_g \mathcal{H}_{d+d^\Lambda}^- = \mathcal{H}_{dd^\Lambda}^-$ (see [22, Proposition 3.24]). This completes the proof of the Theorem. \square

It is helpful to have explicit examples showing clearly the differences between the different cohomologies and $\ker P_J$ discussed above. For this we consider the following examples.

Example 3.3. (cf. [11]) Let $G(c)$ be the connected completely solvable Lie group of dimension 5 consisting of matrices of the form

$$a = \begin{pmatrix} e^{cz} & 0 & 0 & 0 & 0 & x_1 \\ 0 & e^{cz} & 0 & 0 & 0 & y_1 \\ 0 & 0 & e^{cz} & 0 & 0 & x_2 \\ 0 & 0 & 0 & e^{cz} & 0 & y_2 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.1)$$

where $x_i, y_i, z \in \mathbb{R}$ ($i = 1, 2$) and c is a nonzero real number. Then a global system of coordinates x_1, y_1, x_2, y_2 and z for $G(c)$ is given by $x_i(a) = x_i, y_i(a) = y_i$ and $z(a) = z$. A standard calculation shows that a basis for the right invariant 1-forms on $G(c)$ consists of

$$\{dx_1 - cx_1 dz, dy_1 - cy_1 dz, dx_2 - cx_2 dz, dy_2 - cy_2 dz, dz\}. \quad (3.2)$$

Alternatively, the Lie group $G(c)$ may be described as a semidirect product $G(c) = \mathbb{R} \ltimes_\psi \mathbb{R}^4$, where $\psi(z)$ is the linear transformation of \mathbb{R}^4 given by the matrix

$$\begin{pmatrix} e^{cz} & 0 & 0 & 0 \\ 0 & e^{-cz} & 0 & 0 \\ 0 & 0 & e^{cz} & 0 \\ 0 & 0 & 0 & e^{-cz} \end{pmatrix}, \quad (3.3)$$

for any $z \in \mathbb{R}$. Thus, $G(c)$ has a discrete subgroup $\Gamma(c) = \mathbb{Z} \ltimes_{\psi} \mathbb{Z}^4$ such that the quotient space $G(c)/\Gamma(c)$ is compact. Therefore, the forms $dx_i - cx_idz$, $dy_i - cy_idz$ and dz ($i = 1, 2$) descend to 1-forms α_i , β_i and γ ($i = 1, 2$) on $G(c)/\Gamma(c)$.

M. Fernández, V. Muñoz and J. A. Santisteban considered the manifold $M^6(c) = G(c)/\Gamma(c) \times S^1$. Here, there are 1-forms α_1 , β_1 , α_2 , β_2 , γ and η on $M^6(c)$ such that

$$d\alpha_i = -c\alpha_i \wedge \gamma, \quad d\beta_i = -c\beta_i \wedge \gamma, \quad d\gamma = d\eta = 0, \quad (3.4)$$

where $i = 1, 2$ and such that at each point of $M^6(c)$, $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \eta\}$ is a basis for the 1-forms on $M^6(c)$. Using Hattori's theorem, they compute the real cohomology of $M^6(c)$:

$$\begin{aligned} H^0(M^6(c)) &= \langle 1 \rangle, \\ H^1(M^6(c)) &= \langle [\gamma], [\eta] \rangle, \\ H^2(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1], [\alpha_1 \wedge \beta_2], [\alpha_2 \wedge \beta_1], [\alpha_2 \wedge \beta_2], [\gamma \wedge \eta] \rangle, \\ H^3(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \gamma], [\alpha_1 \wedge \beta_2 \wedge \gamma], [\alpha_2 \wedge \beta_1 \wedge \gamma], [\alpha_2 \wedge \beta_2 \wedge \gamma], \\ &\quad [\alpha_1 \wedge \beta_1 \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \eta], [\alpha_2 \wedge \beta_1 \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \eta] \rangle, \\ H^4(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2], [\alpha_1 \wedge \beta_1 \wedge \gamma \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \gamma \wedge \eta], \\ &\quad [\alpha_2 \wedge \beta_1 \wedge \gamma \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta] \rangle, \\ H^5(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \eta] \rangle, \\ H^6(M^6(c)) &= \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta] \rangle. \end{aligned} \quad (3.5)$$

Therefore, the Betti number of $M^6(c)$ are

$$\begin{aligned} b^0 &= b^6 = 1, \\ b^1 &= b^5 = 2, \\ b^2 &= b^4 = 5, \\ b^3 &= 8. \end{aligned} \quad (3.6)$$

We denote by (g, J, ω) be an almost Kähler structure on $M^6(c)$, where we choose

$$g = \alpha_1 \otimes \alpha_1 + \beta_1 \otimes \beta_1 + \alpha_2 \otimes \alpha_2 + \beta_2 \otimes \beta_2 + \gamma \otimes \gamma + \eta \otimes \eta$$

and

$$\omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \gamma \wedge \eta.$$

So J is given by

$$J\alpha_1 = \beta_1, \quad J\alpha_2 = \beta_2, \quad J\gamma = \eta.$$

It is clear that the maps

$$[\omega] : H_{dR}^2(M^6(c); \mathbb{R}) \rightarrow H_{dR}^4(M^6(c); \mathbb{R})$$

and

$$[\omega]^2 : H_{dR}^1(M^6(c); \mathbb{R}) \rightarrow H_{dR}^5(M^6(c); \mathbb{R})$$

are isomorphisms. Thus, $(M^6(c), \omega)$ satisfies the hard Lefschetz property. By simple calculation, we can get

$$\mathcal{Z}_J^- = \text{Span}_{\mathbb{R}}\{\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1\}, \quad (3.7)$$

$$\mathcal{Z}_J^+ = \text{Span}_{\mathbb{R}}\{\alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2, \gamma \wedge \eta\}, \quad (3.8)$$

$$\begin{aligned} \ker P_J &= \text{Span}_{\mathbb{R}}\{\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \\ &\quad \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \alpha_2 \wedge \beta_2 - \gamma \wedge \eta\}. \end{aligned} \quad (3.9)$$

Hence, $\dim \ker P_J = 4 = b^2 - 1$. Of course, J is C^∞ pure and full.

H_{dR}^2	$\alpha_1 \wedge \beta_1, \alpha_1 \wedge \beta_2, \alpha_2 \wedge \beta_1, \alpha_2 \wedge \beta_2, \gamma \wedge \eta$
\mathcal{Z}_J^+	$\alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2, \gamma \wedge \eta$
\mathcal{Z}_J^-	$\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1$
$\mathcal{H}_{d+d^\Lambda}^-$	$\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \alpha_2 \wedge \beta_2 - \gamma \wedge \eta$
$\mathcal{H}_{dd^\Lambda}^-$	$\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \alpha_2 \wedge \beta_2 - \gamma \wedge \eta$
$\ker P_J$	$\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \alpha_2 \wedge \beta_2 - \gamma \wedge \eta$

Table 2. Bases for $H_{dR}^2, \mathcal{Z}_J^+, \mathcal{Z}_J^-, \mathcal{H}_{d+d^\Lambda}^-, \mathcal{H}_{dd^\Lambda}^-$ and $\ker P_J$ of $M^6(c)$.

By the above table, we can see that $\mathcal{H}_{d+d^\Lambda}^-(M^6(c)) = \mathcal{H}_{dd^\Lambda}^-(M^6(c))$. So

$$\mathcal{H}_{d+d^\Lambda}^2(M^6(c)) = \mathcal{H}_{dd^\Lambda}^2(M^6(c)) = \text{Span}_{\mathbb{R}}\{\omega\} \oplus \mathcal{H}_J^- \oplus \mathcal{H}_{J,0}^+.$$

Proposition 3.4. (cf. [11]) *The manifold $M^6(c)$ does not admit Kähler metric.*

Example 3.5. (cf. [22]) Let M be the Kodaira-Thurston nilmanifold defined by taking \mathbb{R}^4 and modding out by the identification

$$(x_1, x_2, x_3, x_4) \sim (x_1 + a, x_2 + b, x_3 + c, x_4 + d - bx_3),$$

where $a, b, c, d \in \mathbb{Z}$. The resulting manifold is a torus bundle over a torus with a basis of cotangent 1-forms given by

$$e_1 = dx_1, \quad e_2 = dx_2, \quad e_3 = dx_3, \quad e_4 = dx_4 + x_2 dx_3.$$

It is well known that Kodaira-Thurston manifold admits no Kähler structure. We take the symplectic form to be

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4.$$

Consider the ω -compatible almost complex structure J given by

$$J(e_1) = e_2, \quad J(e_2) = -e_1, \quad J(e_3) = e_4, \quad J(e_4) = -e_3.$$

Let us denote the compatible metric by

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4.$$

(g, J, ω) is an almost Kähler structure but not Kähler since the almost complex structure J is not integrable. Please see the following table for the relationship between $H_{dR}^2, H_{d+d^\Lambda}^2, H_{dd^\Lambda}^2$ and $\ker P_J$.

H_{dR}^2	$\omega, \quad e_1 \wedge e_2 - e_3 \wedge e_4, \quad e_1 \wedge e_3, \quad e_2 \wedge e_4$
\mathcal{Z}_J^+	$\omega, \quad e_1 \wedge e_2 - e_3 \wedge e_4$
\mathcal{Z}_J^-	$e_1 \wedge e_3, \quad e_2 \wedge e_4$
$H_{d+d^\Lambda}^2$	$\omega, \quad e_1 \wedge e_2 - e_3 \wedge e_4, \quad e_1 \wedge e_3, \quad e_2 \wedge e_4, \quad e_2 \wedge e_3$
$H_{dd^\Lambda}^2$	$\omega, \quad e_1 \wedge e_2 - e_3 \wedge e_4, \quad e_1 \wedge e_3, \quad e_2 \wedge e_4, \quad e_1 \wedge e_4$
$\ker P_J$	$e_1 \wedge e_2 - e_3 \wedge e_4, \quad e_1 \wedge e_3, \quad e_2 \wedge e_4$

Table 3. Bases for $H_{dR}^2, \mathcal{Z}_J^+, \mathcal{Z}_J^-, H_{d+d^\Lambda}^2, H_{dd^\Lambda}^2$ and $\ker P_J$ of M .

By the above table, we can see that (M, ω) does not satisfy the hard Lefschetz property. It is easy to see that the dimension of $\ker P_J$ is equal to $b^2 - 1 = 3$.

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Qiang Tan

School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, China

e-mail: tanqiang1986@hotmail.com

Hongyu Wang

School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, China

e-mail: hywang@yzu.edu.cn

Jiuru Zhou

School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, China

e-mail: zhoujr1982@hotmail.com